A. Cohomology of unitary groups.

Fib. seq.:

$$\begin{tikzcd}
U(n-1) \arrow[r] & U(n) \arrow[r] & S^{2n-1} \\
SU(n-1) \arrow[r] \arrow[u] & SU(n) \arrow[r] \arrow[u] & S^{2n-1} \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

$\underline{n \geq 2}$.

$\underline{\text{Thm}}$. $H^*(U(n), \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_1, x_3, \dots, x_{2n-1})$,

$H^*(SU(n), \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_3, x_5, \dots, x_{2n-1})$,

$H^*(Sp(2n), \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_3, x_7, x_{11}, \dots, x_{4n-1})$.

$\left(\underline{x_i^2 = 0}, \quad x_i x_j = -x_j x_i, \quad i \neq j \right)$.

Proof. Proof by induction.

Base cases:

$U(1) \cong S^1$, $H^*(S^1, \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(x_1)$

$SU(1) = *$,

$SU(2) \cong S^3$,

$Sp(2) \cong S^3$.

Unitary case. Assume true for $n-1$, $n \geq 2$.

$$\boxed{U(n-1) \to U(n) \to S^{2n-1}}.$$

Fiber sequence

$$\pi_i U(n-1) \to \pi_i U(n)$$

iso for $i < 2n-2$,

surj for $i = 2n-2$

$$\left(\pi_i S^{2n-1} = 0 \quad i < 2n-1 \right)$$

$$U(n-1) \to U(n)$$

$(2n-2)$-equivalence

$\Longrightarrow$ $H^i(U(n), \mathbb{Z}) \to H^i(U(n-1), \mathbb{Z})$

is an iso for $i \le 2n-3$.

$$H^*(U(n-1), \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}(\underbrace{x_1, \ldots, x_{2n-3}}))$$

Then lift to classes

$$\tilde{x}_{2i-1} \in H^{2i-1}(U(n), \mathbb{Z}).$$

Remark. Exterior algebras on odd classes are free in the category of graded commutative algebras with no 2-torsion.

$\boxed{y_i y_j = (-1)^{ij} y_j y_i}$

$2y_i^2 = 0$

i odd.

We have lifts $\tilde{x}_1, \tilde{x}_3, \dots, \tilde{x}_{2n-3}$ that generate a subalgebra of $H^*(U(n), \mathbb{Z})$ that looks like $\Lambda_{\mathbb{Z}}(\tilde{x}_1, \dots, \tilde{x}_{2n-3})$.

I also have $y \in H^{2n-1}(U(n), \mathbb{Z})$ pulled back from a generator $H^{2n-1}(S^{2n-1}, \mathbb{Z})$.

By Leray–Hirsch,

$H^*(U(n), \mathbb{Z})$ is a free

$\mathbb{Z}$-module on

polynomials in the $\tilde{x}_{2i-1}$ and y.

$1, x_1, \ldots, x_{2n-3}$

$x_1 x_3, \ldots$

$\ldots$

free

$H^*(S^{2n-1}, \mathbb{Z})$

-module.

In particular, by Leray–Hirsch,

$H^*(U(n), \mathbb{Z})$ is torsion-free.

So, by freeness above,

$$\boxed{\Lambda_{\mathbb{Z}}(\tilde{x}_1, \tilde{x}_3, \ldots, \tilde{x}_{2n-1}, y)} \xrightarrow[\text{L-H}]{\simeq} H^*(U(n), \mathbb{Z}).$$

$\tilde{x}_i^2 = 0$

$y^2 = 0$

$\mathbb{R}$ lin bundle M

Chern classes. $E \to X$ a complex v.b.

There are natural classes

$$c_i(E) \in H^{2i}(X, \mathbb{Z}), \quad i \geq 0,$$

such that

(a) $f^* c_i(E) = c_i(f^* E)$,
$f : Y \to X$,

(b) $c(E \oplus F) = c(E) \cup c(F)$,

$c(E) = 1 + c_1(E) + c_2(E) + \cdots$
(total Chern class),

(c) $c_i(E) = 0, \quad i > \dim_{\mathbb{C}} E$,

(d) $c_1(M_{1,\mathbb{C}})$ is a generator of

$$H^2(BGL_1(\mathbb{C}), \mathbb{Z}) \cong H^2(\mathbb{CP}^\infty, \mathbb{Z}) \cong \mathbb{Z}.$$

Thm. These exist and are unique up to choice

B. Chern classes.

$$\begin{array}{c} E \\ \mathbb{C}\text{-vb} \Big\downarrow \text{rank } d \\ X \end{array}$$

fibers are $\mathbb{C}^d$

$$\begin{array}{c} \mathbb{P}(E) \\ \Big\downarrow \mathbb{C}\mathbb{P}^{d-1}\text{-bundle} \\ X \end{array}$$

fiber over $x \in X$ is the set of lines through 0 in E_x.

<u>Ex</u>. $\underline{\mathbb{C}}^d_X = \mathbb{C}^d \times X$,

$$\mathbb{P}(\underline{\mathbb{C}}^d_X) \cong \mathbb{C}\mathbb{P}^{d-1} \times X.$$

On $\mathbb{P}(E)$, there is a canonical (modular) line bundle $\gamma_1 \subseteq p^* E$

$$\left(\mathbb{P}(E) \xrightarrow{p} X \right). \qquad \left(\mathcal{O}(-1) \right)$$

$$\ell \in \mathbb{P}(E)$$

$$p(\ell) \in X$$

$\ell \subseteq E_x$ rank 1 $\mathbb{C}$-sub vbundle.

$$\{\text{line bundles}\} \longleftrightarrow [-, BU_1] \cong [-, \mathbb{CP}^\infty]$$

$$\cong \underline{\underline{H^2(-, \mathbb{Z})}}.$$

So, let $\boxed{x \in} H^2(\mathbb{P}(E), \mathbb{Z})$ be the coh. class associated to $\underline{\gamma_1}$.

Note: x restricts to a generator of $H^2(\mathbb{CP}^{d-1}, \mathbb{Z})$ for any fiber $\mathbb{CP}^{d-1}$ of $\mathbb{P}(E) \longrightarrow X$.

$\Rightarrow$ $1, x, x^2, \ldots, x^{d-1}$ restrict to generate (additively) the cohomology of the fiber.

So, Leray–Hirsch applies

$\Rightarrow$ $\underline{H^*(\mathbb{P}(E), \mathbb{Z})}$ is free over $\underline{H^*(X, \mathbb{Z})}$ on the classes $\underline{1, x, \ldots, x^{d-1}}$.

$$\boxed{x^d - c_1(E)x^{d-1} + c_2(E)x^{d-2} - \cdots + (-1)^d c_d(E)\cdot 1 = 0}$$

$H^{2d}(\mathbb{P}(E),\mathbb{Z})$ 2d-2

H^2 $c_i(E) \in H^{2i}(X,\mathbb{Z})$.

Note. If E has $d=1$.

$$\mathbb{P}(E) \xrightarrow[p]{\cong} X.$$

$x \in H^2(X,\mathbb{Z})$ $\qquad x - c_1(E) = 0.$

$\updownarrow$

E $\qquad$ I.e., $c_1(E) = x$.

So, γ_1 on $\mathbb{CP}^\infty \simeq BU_1$, get that $c_1(\gamma)$ generates $H^2(\mathbb{CP}^\infty,\mathbb{Z})$.

D. Splitting principle.

<u>Proposition</u>. X is a paracompact Hausdorff space, $E \to X$ a complex rank d v.b., then there is a map $Y \xrightarrow{f} X$ s.t. (a) $H^*(X;\mathbb{Z}) \overset{f^*}{\hookrightarrow} H^*(Y;\mathbb{Z})$ is injective,

(b) $f^*E \cong L_1 \oplus \cdots \oplus L_d$ a sum of line bundles.

<u>Proof</u>. $\mathbb{P}(E) \xrightarrow{p} X$

$$\gamma_1^E \hookrightarrow p^*E \Rightarrow p^*E \cong \gamma_1^E \oplus F.$$

$$\mathbb{P}(F) \xrightarrow[q]{} \mathbb{P}(E) \longrightarrow X$$

$$\gamma_1^F \hookrightarrow q^*F \Rightarrow q^*F \cong \gamma_1^F \oplus G$$

i $\quad\square$.

Remark. This gives a flag bundle.

$$F\ell(E) \longrightarrow X$$

$$\boxed{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq E}$$

$\dim V_i = i$.

Proof that $c(E \oplus F) = c(E) \cup c(F)$.

Use splitting principal to reduce to the case where E and F are sums of line bundles.

Now: have to show that

$$c(\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_d) = c(\mathcal{L}_1) \cup \cdots \cup c(\mathcal{L}_d)$$

$$\underset{(c)}{=} (\underline{1 + c_1}(\mathcal{L}_1)) \cup \cdots \cup (1 + c_1(\mathcal{L}_d))$$

$$E = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_d$$

$$\mathbb{P}(E) \xrightarrow{p} X$$

$$0 \to \gamma_1^E \to p^+ E \to \sigma \to 0$$

$$\gamma_1^{E,\vee} \otimes$$

$$0 \to \underline{\mathbb{C}}^1 \xrightarrow{s} \gamma_1^\vee \otimes p^+ E \to \gamma_1^\vee \otimes \sigma \to 0$$

Nowhere vanishing section.

$$\xrightarrow{s_i} \gamma_1^\vee \otimes p^* \mathcal{L}_i$$

s_i is a section of $\gamma_1^\vee \otimes p^* \mathcal{L}_i$.

$$Z_i = \text{zero locus} \subset \mathbb{P}(E)$$

$$U_i = \mathbb{P}(E) \setminus Z_i,$$

where s_i is non-zero.

$\Longrightarrow$ $c_1(\gamma_1^\vee \otimes p^* \mathcal{L}_i)$ restricts to zero on U_i.

$\Longrightarrow$ so it lifts to

$$H^2(\mathbb{P}(E), U_i, \mathbb{Z}).$$

$$\Longrightarrow \quad c_1(\gamma_1^\vee \otimes p^* \mathcal{L}_1) \cup \dots \cup c_1(\gamma_1^\vee \otimes p^* \mathcal{L}_d)$$

$$\in H^{2d}\Big(\mathbb{P}(E), \underbrace{\bigcup_{i=1}^{d} U_i}_{= \mathbb{P}(E)}, \mathbb{Z}\Big) = 0$$

$$\Longrightarrow \prod_{i=1}^{d} \left(c_1(\gamma_1^\vee) + p^* c_1(L_i) \right) = 0$$

expanding out and using that

$$c_1(\gamma_1^\vee) = -c_1(\gamma_1)$$

$$\Longrightarrow (-1)^d c_1(\gamma_1)^d + (-1)^{d-1} c_1(\gamma_1)^{d-1} \left(p^*(c_1(L_1)) + \cdots + p^* c_1(L_d) \right) + \cdots + \prod_{i=1}^{d} p^* c_1(L_d) = 0.$$

$$\Longrightarrow c(L_1 \oplus \cdots \oplus L_d) = c(L_1) \cup \cdots \cup c(L_d).$$

Rem. The chern classes are the elementary symmetric functions in c_1's of the line bundles.